Counterfeiting substitute media-of-exchange: a threat to monetary systems*

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Abstract

One justification for cash-in-advance equilibria is the assumption that the counterfeiting of money is impossible, while the counterfeiting of higher-return substitute media-of-exchange is costless. It is shown that this justification is not robust to replacing costless by costly and placing the analysis within a signaling-game framework in which the intuitive criterion is invoked. In particular, if the cost of counterfeiting the substitutes is small, then there is no monetary equilibrium. Therefore, the counterfeiting of substitutes can be a threat to monetary systems. This result provides a new rationale for legal restrictions that prohibit trades using higher-return substitute media-of-exchange.

Keywords: Counterfeiting; cash-in-advance; monetary system; signaling games

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1 Introduction

Most countries devote substantial resources to discouraging counterfeiting of their currencies and devote far fewer resources to discouraging counterfeiting of alternative assets that can potentially serve as media-of-exchange. Consistent with that, there is a literature — e.g., Freeman [4] and Lester *et. al.* [6] — that justifies cash-in-advance equilibria in the presence of higher-return assets through the assumption that money is not subject to a recognizability problem (no counterfeiting of money) and that substitute higher-return media-of-exchange (hereafter called *bonds*) are subject to a huge recognizability problem (costless and unlimited counterfeiting of bonds). I show that existence of the cash-in-advance equilibrium is not robust to (i) assuming that the counterfeiting of bonds is costly, even if the cost is tiny, and (ii) placing the analysis within a signaling-game framework and invoking the intuitive criterion proposed by Cho and Kreps [3].

The reasoning is straightforward. First, under the assumptions made — and, quite plausibly — counterfeit bonds have value only if there is a pooling equilibrium in which they masquerade as genuine bonds in transactions prior to maturity. But, as shown here, there is no pooling equilibrium (that satisfies the intuitive criterion). Hence, there is no equilibrium with counterfeiting. There may be an equilibrium without counterfeiting, but only if the cost of counterfeiting bonds is sufficiently high. As might be expected, this equilibrium is a bond-in-advance equilibrium, not a cash-in-advance equilibrium. More interestingly, if the cost of counterfeiting bonds is sufficiently small, even for a small fraction of the population, then there is no monetary equilibrium. This last result resembles what Nosal and Wallace [8] find: in a model in which money can be counterfeited, the counterfeiting of money can be a threat to the monetary system.[1] Here, the counterfeiting of substitute media-of-exchange can be a threat to the monetary system.

[1]Nosal and Wallace [8] seems to be the first paper on counterfeiting that applies the intuitive criterion. Other papers with private information about the quality of assets held use pooling equilibria without invoking it. See, for example, Green and Weber [5], Velde *et. al.* [10], Williamson [12], and Williamson and Wright [13].

2 A partial equilibrium model

Because the main result is nonexistence of an euilibrium, it is sufficient to use a partial equilibrium model. Time is discrete with two stages at a date. Stage 1 has portfolios decisions, while stage 2 has pairwise meetings with production and consumption.

There is a continuum of agents, each of whom begins a date with some divisible money. Half are buyers and half are sellers and, in addition, some fraction of the buyers have a technology that allows them to counterfeit bonds at cost c (with zero marginal cost). Then the agents make portfolio choices: they can buy (genuine) divisible discount bonds at price $p \in (0, 1]$ in terms of money. One unit of bond is a transferable claim to one unit of money at the end of the date.[2] Buyers with the counterfeiting technology can choose to counterfeit. Those actions complete stage 1. At the end of stage 1, each buyer is characterized by (m, b, t), where $m \in M = [0, \overline{m}]$ is the money holding, $b \in B = [0, \overline{b}]$ is the (genuine) bond holding, and $t \in \{h, l\}$, where h (high) means that the person has not produced counterfeit bonds and l (low) means that the person has produced counterfeit bonds and has incurred cost c; each seller is characterized by nominal wealth, denoted $z \in Z = [0, \overline{m} + \overline{b}]$. The stage-1 buyer decisions give rise to a distribution φ over $M \times B \times \{h, l\}$, the set of buyer types. Similarly, the stage-1 seller decisions give rise to a distribution π over Z, the set of seller types. Both φ and π are common knowledge among the agents.

At stage 2, there are random pairwise meetings, with each pair consisting of a buyer and a seller. I assume that the wealth of the seller is known, but the buyer type is private information. In a meeting between a buyer of type (m, b, t) and a seller with wealth z, the buyer makes an offer, (x_m, x_b, y), where $x_m \in [0, \overline{m}]$ and $x_b \in [0, \overline{b}]$ denote the proposed money and bond transfers to the seller, respectively, and $y \in Y = [0, \infty)$ denotes the amount of good to be transferred to the buyer. If the seller says yes and the buyer's type is h, then the seller's payoff is $-y + V_s(z + x_m + x_b)$ and the buyer's payoff is $u(y) + V_b(m + b - x_m - x_b)$. If the seller says yes and the buyer's type is l, then the seller's

[2] Aside from the counterfeiting, this resembles the set-up in Zhu and Wallace [11].

payoff is $-y + V_s(z + x_m)$ and the buyer's payoff is $u(y) + V_b(m + b - x_m) - c$.[3] If the seller says no, then there is no trade, which has the obvious payoffs. In particular, the payoff to a type-l buyer is $u(0) + V_b(m + b) - c$. Built into these payoffs is the assumption that counterfeit bonds are worthless after stage 2 — because the issuer can perfectly distinguish genuine bonds from counterfeits. I assume that u, V_s, and V_b are strictly increasing and concave, and that $u'(0) = \infty$.

Because the wealth level of the seller is common knowledge in a meeting, each level z defines a proper subgame which has the form of a signaling game. For a given z, a buyer's strategy is a function $s_z : M \times B \times \{h, l\} \to M \times B \times Y$, which assigns to each buyer type an offer and is subject to the obvious feasibility constraints that limit asset transfers. A seller *strategy* and *belief* is a function $(r_z, \psi_z) : M \times B \times Y \to \{yes, no\} \times [0, 1]$, which assigns to each buyer offer observed by the seller a response to the offer and a probability that the buyer's type is l.

The definition of equilibrium I use is in the spirit of perfect Bayesian equilibrium. It differs from perfect Bayesian equilibrium because the set of strategies is infinite. The challenge in such cases is to specify a reasonable restriction on seller beliefs in equilibrium. My approach, which is similar to that in Ramey [9], is to restrict the support of the belief without imposing quantitative restrictions when Bayes rule is not applicable. Given z, a buyer offer (x_m, x_b, y) is an *equilibrium offer* if there is some buyer type $(m, b, t) \in \text{supp}(\varphi)$ such that $s_z(m, b, t) = (x_m, x_b, y)$, where $\text{supp}(\varphi)$ denotes the support of φ. Similarly, the support of π is denoted by $\text{supp}(\pi)$. Let (s, r, ψ) denote the family $\{(s_z, r_z, \psi_z)\}_{z \in Z}$. Then we have the following partial equilibrium definition, partial because it takes as given the price p, the distributions φ and π, and the value functions V_b and V_s.

Definition 1. Given distributions φ and π, (s, r, ψ) is a *stage-2 partial equilibrium* if the following conditions hold:

(i) for each $z \in \text{supp}(\pi)$, s_z is a best response to r_z;

[3] Although a buyer's type has three components, only the third — whether the buyer holds counterfeit bonds or not — affects the seller's payoff. For this reason, the label type-t buyer is useful and means a buyer of type (m, b, t) for some portfolio (m, b).

(ii) for each $z \in \text{supp}(\pi)$ and each buyer offer (x_m, x_b, y), $r_z(x_m, x_b, y)$ is a best response to (x_m, x_b, y) given $\psi_z(x_m, x_b, y)$;

(iii) for each $z \in \text{supp}(\pi)$ and for any equilibrium offer (x_m, x_b, y) given z, $\psi_z(x_m, x_b, y)$ is calculated via Bayes' rule whenever possible and if not, then

$$|\delta_t^h - \psi_z(x_m, x_b, y)| > 0 \text{ iff } \{(m, b, t) : s_z(m, b, t) = (x_m, x_b, y)\} \cap \text{supp}(\varphi) \neq \emptyset,$$

where $\delta_h^h = 1$ and $\delta_l^h = 0$.

The last part of condition (iii) is a mild restriction on $\psi_z(x_m, x_b, y)$ on equilibrium paths when Bayes' rule cannot be applied. In addition to the restrictions in Definition 1, there is a refinement literature that imposes restrictions on beliefs for off-equilibrium actions. I follow most of the literature and adopt the intuitive criterion proposed by Cho and Kreps [3].

Before turning to the analysis of the model, several comments are in order about it. Among the many assumptions made, some are mainly for expositional ease, while others are more significant.

The first group includes almost everything about the environment. For example, the timing of the assignment of people to buyer and seller status is not essential. Nor is the assumption that every meeting is between a buyer and seller, or the special assumption about the distribution of counterfeiting costs. The assumption that there is only one round of trade prior to maturity can be relaxed to allow for any finite number of rounds of such trade and, perhaps, for an infinite number. And the nonexistence of a pooling equilibrium depends only on the assumption that ending trade with a counterfeit bond is worth less than ending with a genuine bond. Nor do the results depend on the assumption that the marginal cost of producing counterfeits is zero.

The second group is about the game played and the equilibrium concept. The significant assumptions are the formulation of the stage-2 game as a signaling game and the use of the intuitive criterion. However, the assumption that the seller types are common knowledge can be dropped. If seller wealth is private information, then a buyer offer consists of a menu, each item of which is intended for a different seller. All the analysis below can be modified to permit this and I suspect that all the main results still hold. Obviously, the nonexistence result holds for refinements of the intuitive criterion, but may not hold for alternatives to it.[4] One route to generalization beyond the signaling game formulation is to use a notion of the (pairwise) core under asymmetric information. However, as yet, there is no widely accepted notion of that core.

3 No counterfeiting

As I now show, if an equilibrium that satisfies the intuitive criterion exists and if $c > 0$, then there is no counterfeiting. The proof is by contradiction. The first step is to show that a necessary condition for such an equilibrium is pooling, which is defined as follows:

Definition 2. A pooling equilibrium (in bonds) is a stage-2 partial equilibrium (s, r, ψ) such that there is some equilibrium offer (x_m, x_b, y) and some wealth level $z \in \text{supp}(\pi)$ satisfying (i) $r_z(x_m, x_b, y) = yes$; (ii) $0 < \psi_z(x_m, x_b, y) < 1$; and (iii) $x_b > 0$.

The next lemma shows that if there is counterfeiting, then the stage-2 partial equilibrium is a pooling equilibrium.

Lemma 1. *There is counterfeiting in a monetary equilibrium only if the stage-2 partial equilibrium is a pooling equilibrium.*

The proof, which appears in the appendix, would be easy if the type space were finite, as would be the case if money and bonds came in indivisible units.[5] Obviously, the existence of counterfeiting implies that there are type-l buyers making the same offers as some type-h buyers, but the challenge with infinite type-spaces lies in the possibility that the seller may take this as a negligible event and assign probability zero to it. This difficulty is resolved in the proof via a measure-theoretic argument.

[4] One alternative is Mailath *et. al.* [7]. For an epistemic justification for the intuitive criterion in terms of rationality, see Battigalli and Siniscalchi [1].

[5] If money and bonds were indivisible, then it would be necessary to introduce lotteries, as is done in [8].

Now, in order to show that there is no counterfeiting, it is sufficient to show that there is no pooling equilibrium. The proof is a slight extension of the standard demonstration that the single-crossing property holds in this model. In particular, given a candidate pooling equilibrium, there exists a deviating offer by a type-h buyer. The offer has smaller proposed trades of both the good and bonds. Therefore, it can only come from a type-h buyer and, as a consequence, is accepted by the seller. The new part of the argument excludes the possibility that the deviating offer is offered in equilibrium by some other buyer. This possibility does not arise in models with only two types, as is the case in [8].

Theorem 1. *There is no pooling equilibrium that satisfies the intuitive criterion.*

Proof. Suppose that there exists such an equilibrium (s, r, ψ) and suppose that $(x_m, x_b, y) = s_z(m, b, h)$ satisfies conditions (i)-(iii) in Definition 2 with $z \in \text{supp}(\pi)$. Let $\psi_z(x_m, x_b, y) = \alpha \in (0, 1)$.

I first show there exist $\epsilon > 0$ and $\epsilon' > 0$ such that

$$u(y - \epsilon') + V_b(m + b - x_m - x_b + \epsilon) > u(y) + V_b(m + b - x_m - x_b) \tag{1}$$

and

$$-y + \epsilon' + V_s(z + x_m + x_b - \epsilon) > -y + (1 - \alpha)V_s(z + x_m + x_b) + \alpha V_s(z + x_m) \tag{2}$$

hold.

Inequality (1) says that the offer $(x_m, x_b - \epsilon, y - \epsilon')$ gives higher payoff to the buyer of type (m, b, h) and inequality (2) ensures that the seller with wealth level z will accept the offer, provided that it is from a type-h buyer.

Inequality (2) is equivalent to $\epsilon' > -f(\epsilon)$, where

$$\begin{aligned} f(\eta) &= \alpha[V_s(z + x_m + x_b - \eta) - V_s(z + x_m)] \\ &-(1 - \alpha)[V_s(z + x_m + x_b) - V_s(z + x_m + x_b - \eta)]. \end{aligned}$$

Because

$$f(0) = \alpha[V_s(z + x_m + x_b) - V_s(z + x_m)] > 0,$$

and V_s is continuous and strictly increasing, we can choose $\epsilon > 0$ so that $f(\epsilon) > 0$. If so, then inequality (2) holds for any $\epsilon' \geq 0$. Then, for any such ϵ, choose $\epsilon' > 0$ so that

$$u(y) - u(y - \epsilon') < V_b(m + b - x_m - x_b + \epsilon) - V_b(m + b - x_m - x_b). \tag{3}$$

Because V_b is strictly increasing and u is continuous, this can be done.

Now I show in two steps that $(x_m, x_b - \epsilon, y - \epsilon')$ is a non-equilibrium offer and that it leads to a worse payoff for any type-l buyer than his equilibrium payoff (regardless of the seller's response to this deviation). The first step shows that it is not made by a type-l buyer — either as an equilibrium offer or as a deviation; the second step, which depends on the first step, shows that it is not an equilibrium offer by any type-h buyer.

Step 1. It is, of course, enough to deal with type-l buyers whose money holding is at least x_m. So consider an arbitrary type-l buyer; namely type (m', b', l), with $m' \geq x_m$ and $s_z(m', b', l) = (x'_m, x'_b, y')$. I consider two cases.

Case (a): $m' > 0$. Then there exists a feasible offer $(x''_m, 0, y'')$ satisfying

$$u(y'') + V_b(m' + b' - x''_m) > u(0) + V_b(m' + b') \tag{4}$$

and that is accepted by the seller. That is, some money trade, offering some amount of money for some amount of the good, is accepted by the seller and gives the buyer a higher payoff than no trade. This follow from $u'(0) = \infty$ and concavity of V_b and V_s.[6] Hence, such a type-l buyer does not make an unacceptable offer.

It follows that $s_z(m', b', l) = (x'_m, x'_b, y')$ is accepted. Therefore, its payoff is

$$\begin{aligned} u(y') + V_b(m' + b' - x'_m) &\geq \max\{u(y) + V_b(m' + b' - x_m), u(y'') + V_b(m' + b' - x''_m)\} \quad (5) \\ &> \max\{u(y - \epsilon') + V_b(m' + b' - x_m), u(0) + V_b(m' + b')\}. \end{aligned}$$

Inequality (5) implies that the buyer receives a strictly worse payoff by offering $(x_m, x_b - \epsilon, y - \epsilon')$ (whether the seller accepts it or not) than the equilibrium payoff. (The first

[6]This is the only place where $u'(0) = \infty$ and concavity of the value functions are used. These are sufficient, but obviously not necessary. All that is required is that some money trade is better than no trade.

inequality in (5) follows from the fact that s is an equilibrium strategy and that the offers (x_m, x_b, y) and $(x''_m, 0, y'')$ are feasible for the buyer of type (m', b', l), while the second inequality in (5) follows from strict monotonicity of u and inequality (4).)

Case (b): $m' = 0$. Then, $x_m = 0$. In this case, using only strict monotonicity of u, $(0, x_b, y)$ is a strictly better offer (which is accepted by the seller) than $(0, x_b - \epsilon, y - \epsilon')$ (whether it is accepted or not).

Step 2. Now I show that $(x_m, x_b - \epsilon, y - \epsilon')$ is not an equilibrium offer made by any type-h buyer. Suppose it is. Because (x_m, x_b, y) is accepted by the seller, inequality (1) implies that any buyer for whom (x_m, x_b, y) is feasible obtains a higher payoff by making the offer $(x_m, x_b - \epsilon, y - \epsilon')$ provided that the seller accepts it. Step 1 and condition (iii) in Definition 1 imply that $\psi_z(x_m, x_b - \epsilon, y - \epsilon') = 0$. Then (2) implies such acceptance. This contradicts the assumption that (s, r, ψ), in which (x_m, x_b, y) is an equilibrium offer by a type-h buyer, is an equilibrium.

I have now shown that $(x_m, x_b - \epsilon, y - \epsilon')$ satisfies all the conditions required for showing that (s, r, ψ) fails the intuitive criterion. The offer $(x_m, x_b - \epsilon, y - \epsilon')$ is a non-equilibrium offer that can come only from a type-h buyer. By (2), it is accepted by the seller, and, by (1), the buyer of type (m, b, h) prefers it to (x_m, x_b, y). $\square$

Having shown that there is no equilibrium with counterfeiting, I next consider whether there are equilibria without counterfeiting.

4 Does monetary equilibrium exist?

I structure the discussion in terms of three cases for c and p. I start with the general case: $p < 1$ and $c > 0$. Then I consider $p = 1$ and $c > 0$. And, lastly, mainly because this case is in the literature, I consider $p \leq 1$ and $c = 0$.

Case $p < 1$ and $c > 0$. Here, $c > 0$ implies that there is no counterfeiting. Therefore, bonds offered at stage 2 are regarded as genuine. It follows that no one leaves stage 1 with money and that all stage-2 trade is conducted with bonds. However, this is an equilibrium only if c is sufficiently large. If not, then any buyer with the counterfeiting technology will want to counterfeit, which contradicts the conclusion that there is no counterfeiting in equilibrium. Hence, unless c is sufficiently large, the assumption that V_b and V_s are strictly increasing and concave is contradicted. In that sense, the counterfeiting of bonds threatens the monetary system.

Case $p = 1$ and $c > 0$. Again, there is no counterfeiting in equilibrium. But now there are equilibria. There is an equilibrium in which bonds are ignored (none are purchased at stage 1) and there are equilibria in which some bonds are bought at stage 1, but the amount traded at stage 2 is so small that counterfeiting is unprofitable.

Case $p \leq 1$ and $c = 0$. In this case, Theorem 1 continues to hold. But Lemma 1 does not. Hence, there may be an equilibrium in which those with the counterfeiting technology counterfeit, the counterfeits are not traded, and only money is used at stage 2. However, $c = 0$ is a special and unappealing case.

5 Concluding remarks

So far, nothing has been said about who offers bonds and the determination of p. Governments rarely offer securities which can easily substitute for their monies. Therefore, let's consider the possibility that the transferable bonds are offered by intermediaries. In particular, suppose that the government at stage 1 makes available one-period discount bonds that are *not* suitable to be traded in pairwise meetings — perhaps, because they are in large denominations or are book-entry securities. The transferable bonds in the model could arise from the following intermediation activity: an intermediary holds as assets bonds issued by the government and offers one-period transferable bonds that are designed to be traded in pairwise meetings. If $p' < 1$ is the price in terms of money at which the government offers its bonds, then the intermediary can offer its transferable bonds at $p \in (p', 1)$ and earn revenue proportional to $p - p'$. But if such intermediary bonds are subject to being counterfeited, then all the results above apply. In particular, unless the cost of counterfeiting the intermediary bonds is sufficiently high, there is no equilibrium with such intermediation.

This does not, of course, imply that there is an equilibrium in which no one chooses to enter that intermediation business. It does, however, imply that a legal restriction that prohibits such intermediation does not eliminate equilibria and is consistent with a monetary equilibrium. In that sense, the above analysis provides a new rationale for such a legal restriction.

6 Appendix: Proof of Lemma 1

Let $s_z(m,b,t) = (s_{z,x_m}(m,b,t), s_{z,x_b}(m,b,t), s_{z,y}(m,b,t))$. Suppose that there is counterfeiting in equilibrium; i.e., $q_l = \varphi(M \times B \times \{l\}) > 0$. Notice that $q_l < 1$ in equilibrium. Let φ_t denote the conditional distribution of φ over $M \times B \times \{t\}$, $t = h, l$. Consider a buyer with type (m,b,l) that is in the support of φ. Let

$$A_{m,b} = \{z \in \text{supp}(\pi) : s_{z,x_b}(m,b,l) > 0,\ \psi_z(s_z(m,b,l)) < 1, r_z(s_z(m,b,l)) = yes\}.$$

This is the set of seller types such that the buyer of type (m,b,l) offers a positive amount of bonds to them, they assign positive probability that the offer comes from a type-h buyer, and they respond *yes*. We first show that $\pi(A_{m,b}) > 0$.

For each $z \notin A_{m,b}$, there are two possibilities: either $z \notin \text{supp}(\pi)$, or $z \in \text{supp}(\pi)$, but (a) the seller rejects the offer $s_z(m,b,l)$; or (b) the seller accepts the offer and $s_{z,x_b}(m,b,l) = 0$; or (c) $\psi_z(s_z(m,b,l)) = 1$ and the seller accepts the offer. If $\pi(A_{m,b}) = 0$, then the buyer of type (m,b,l) is better off by not counterfeiting and changing the offers as follows: for z's not in the support, or for z's in the support but belonging to cases (a) or (b), no change is needed; for z's in the support and belonging to case (c), change the offer to $(s_{z,x_m}(m,b,l), 0, s_{z,y}(m,b,l))$, which the seller will accept. This deviation cannot happen because (s,r,ψ) is an equilibrium. Hence, $\pi(A_{m,b}) > 0$.

If there is some $(m,b) \in \text{supp}(\varphi_l)$, and some $z \in A_{m,b}$ with $\psi_z(s_z(m,b,l)) > 0$, then (s,r,ψ) is a pooling equilibrium. So suppose that for all $(m,b) \in \text{supp}(\varphi_l)$, and for all $z \in A_{m,b}$, $\psi_z(s_z(m,b,l)) = 0$. Now I show that this leads to a contradiction.

It follows from condition (iii) in Definition 1 that for all $(m, b) \in \mathrm{supp}(\varphi_l)$ and for all $z \in A_{m,b}$,

$$\varphi_h(\{(m', b') : s_z(m', b', h) = s_z(m, b, l)\}) > 0, \tag{6}$$

and

$$\varphi_l(\{(m', b') : s_z(m', b', l) = s_z(m, b, l)\}) = 0. \tag{7}$$

For each $z \in A = \bigcup_{(m,b)\in\mathrm{supp}(\varphi_l)} A_{m,b}$, let

$$X_z = \{(x_m, x_b, y) : (x_m, x_b, y) = s_z(m, b, l) \text{ for some } (m, b) \text{ such that } z \in A_{m,b}\}.$$

The set X_z consists of equilibrium buyer offers to sellers with wealth z that include counterfeit bonds. Let

$$T = \{(m, b, z) : (m, b) \in \mathrm{supp}(\varphi_l), z \in A_{m,b}\}.$$

This consists of portfolio types of type-l buyers and wealth levels of sellers such that counterfeits are offered and accepted.

There are two valid ways to calculate the measure of the set T. I show that it is zero by one method and positive by the other. That is a contradiction. Let $\varphi_l \otimes \pi$ denote the product measure of φ_l and π over $(M \times B) \times Z$.

First, it can be calculated via the sets X_z. It is easy to see that

$$T \subseteq \{(m, b, z) : s_z(m, b, l) \in X_z, z \in A\} = U.$$

Now, the measure of U can be computed as follows: first compute the measure of the set $\{(m, b) : s_z(m, b, l) \in X_z\}$ for each $z \in A$, then integrate these measures over z. Inequality (6) implies that for each offer in X_z, the set of portfolio types of type-h buyers who offer it has positive measure; i.e., for each $z \in A$ and for each $(x_m, x_b, y) \in X_z$, $\varphi_h(B^z_{x_m,x_b,y}) > 0$, where

$$B^z_{x_m,x_b,y} = \{(m, b) : s_z(m, b, h) = (x_m, x_b, y)\}.$$

The set $B^z_{x_m,x_b,y}$ consists of type-h buyers who make the offer (x_m, x_b, y) to sellers with wealth z. For different offers (x_m, x_b, y) and (x'_m, x'_b, y') in X_z, the sets $B^z_{x_m,x_b,y}$ and $B^z_{x'_m,x'_b,y'}$ are disjoint. Because

$$\varphi_h(\bigcup_{(x_m,x_b,y)\in X_z} B^z_{x_m,x_b,y}) \leq 1 \text{ and } \varphi_h(B^z_{x_m,x_b,y}) > 0 \text{ for all } (x_m, x_b, y) \in X_z,$$

the set X_z is at most countably infinite.[7]

Now, (7) implies that for each $z \in A$ and for each $(x_m, x_b, y) \in X_z$,

$$\varphi_l(\{(m, b) : s_z(m, b, l) = (x_m, x_b, y)\}) = 0.$$

Hence,

$$\begin{aligned}(\varphi_l \otimes \pi)(T) \leq (\varphi_l \otimes \pi)(U) &= \int_{z\in A} \varphi_l(\{(m, b) : s_z(m, b, l) \in X_z\})d\pi(z) \\ &= \int_{z\in A} \sum_{(x_m,x_b,y)\in X_z} \varphi_l(\{(m, b) : s_z(m, b, l) = (x_m, x_b, y)\})d\pi(z) = 0.\end{aligned} \tag{8}$$

Second, it can be calculated directly. Because $\pi(A_{m,b}) > 0$ for all $(m, b) \in \text{supp}(\varphi_l)$,

$$(\varphi_l \otimes \pi)(T) = \int_{(m,b)\in\text{supp}(\varphi_l)} \pi(A_{m,b})d\varphi_l(m, b) > 0. \tag{9}$$

Now, (8) and (9) contradict each other. $\square$

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